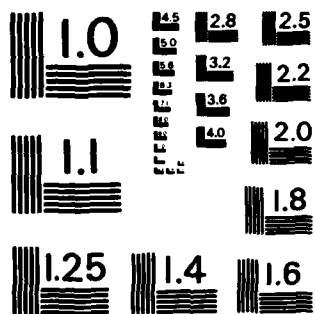END
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THREE POSITION ESTIMATION PROCEDURES

by

R. N. FORREST
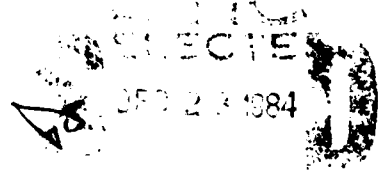
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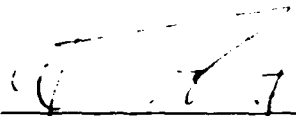
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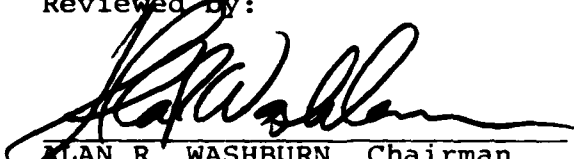

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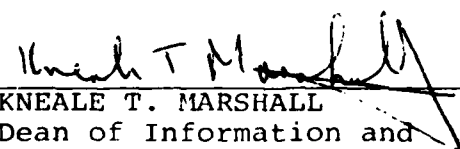
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report describes three position estimation procedures. The first is for estimates based on bearings taken on or from stations with uncertain locations. The second is for use with two or more lines of position. The third is for combining estimates from different sources.

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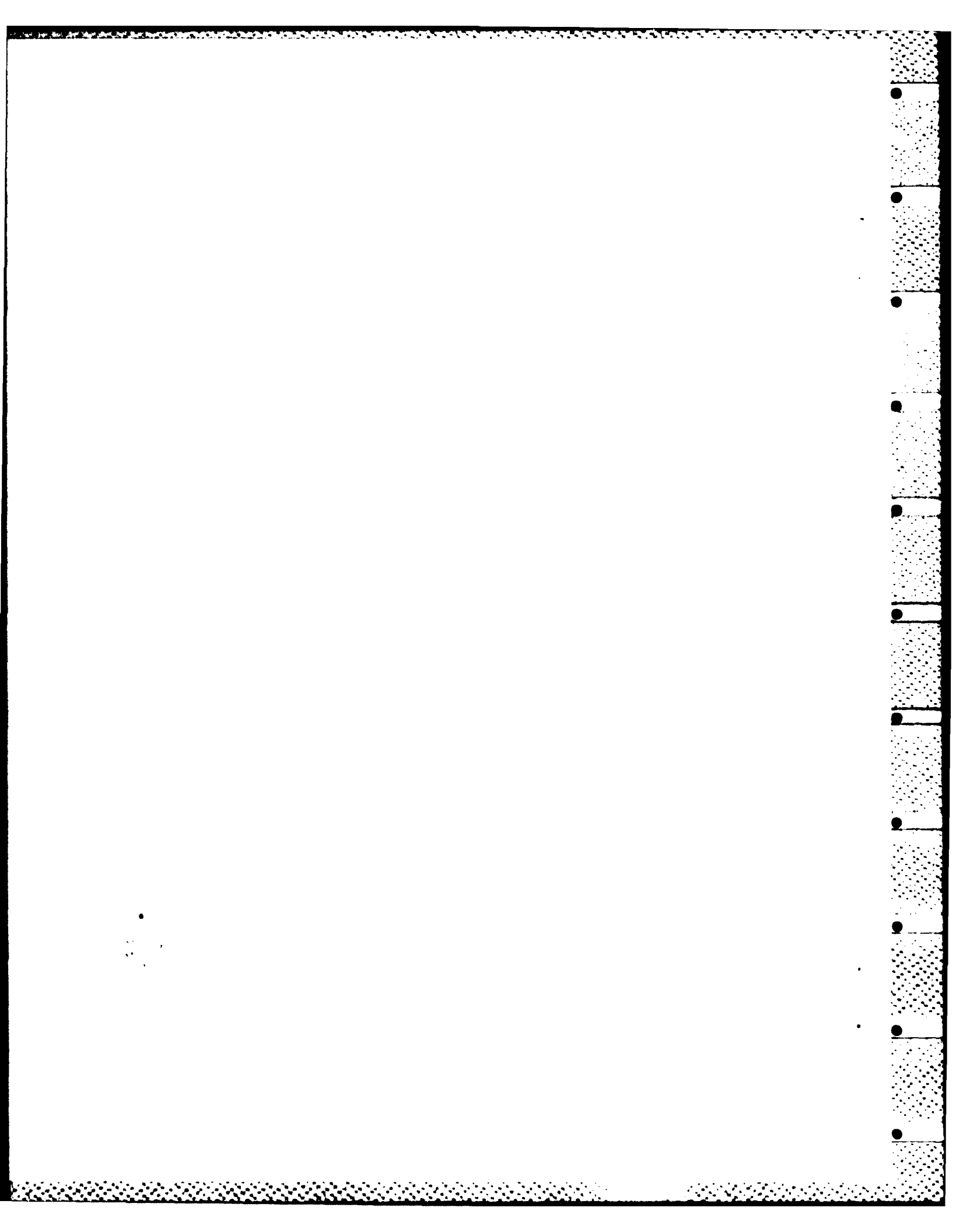
I. Introduction

This report describes position estimation procedures that are based on models which relate positional uncertainty to various measurement and estimation errors. The procedures were developed to be used in analyses that relate the effect of positional uncertainty on tactical performance through such factors as weapon accuracy as well as to be used operationally.

In the models, positions are on a plane surface (flat earth model). Because of this condition, the models are not intended for use in situations where the earth's figure is significant. In addition, positional errors are determined by independent normally distributed random variables with known means, variances and covariances. The support for this condition, other than its mathematical convenience, is that it has been used by others, for example, see Reference 1. To use the models, one is required to specify the means, variances and covariances of the error random variables. For these models, this can be done by specifying the systematic errors (biases) and the error magnitudes (standard deviations).

The procedure that is described in Section II relates position estimates that are based on bearings on or from stations to station position uncertainty. One application for the model is the analysis of the effect of sonobuoy position uncertainty on position estimates determined with passive directional sonobuoys.

The procedure that is described in Section III relates position estimates that are based on lines of position to line of

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position uncertainty. In addition to estimating the uncertainty in position estimates based on celestial observations, the procedure could be used to determine error ellipses for LORAN fixes if the standard deviation values required by the procedure could be obtained.

The procedure that is described in Section IV can be used to combine position estimates from various sources. The procedure which is based on conditions that should not be too restrictive in most cases provides both a composite position estimate and error ellipse.

II. Station Position Uncertainty and Position Estimates

The procedure that is described in this section relates position estimates that are based on bearings on or from stations to station position uncertainty. The procedure is based on a model that is an extension of one that is described on Reference 2. The model is defined as follows: Each station position error is determined by an independent bivariate normal distribution with a zero mean vector and a known covariance matrix. Observed bearing lines on or from a station are parallel to true bearing lines. The distance of each observed bearing line from its true bearing line is determined by an independent normal distribution with a zero mean and a standard deviation σ.

Because a station's position error is determined by a bivariate normal distribution, the perpendicular distance s between a line at the assumed location of an observed bearing line and the observed bearing line is determined by a normal distribution. The relation between the bivariate normal distribution that describes the station's position uncertainty is indicated in Figure 1. In the figure, the positive y-axis direction is north, the positive x-axis direction is east, and the origin of the coordinate system is at the assumed station position. The x'y'-coordinate system is oriented so that the positive x'axis is coincident with the major axes of the elliptical contours of the bivariate normal distribution that determines the station position error and so that the bearing δ of the positive x'-axis
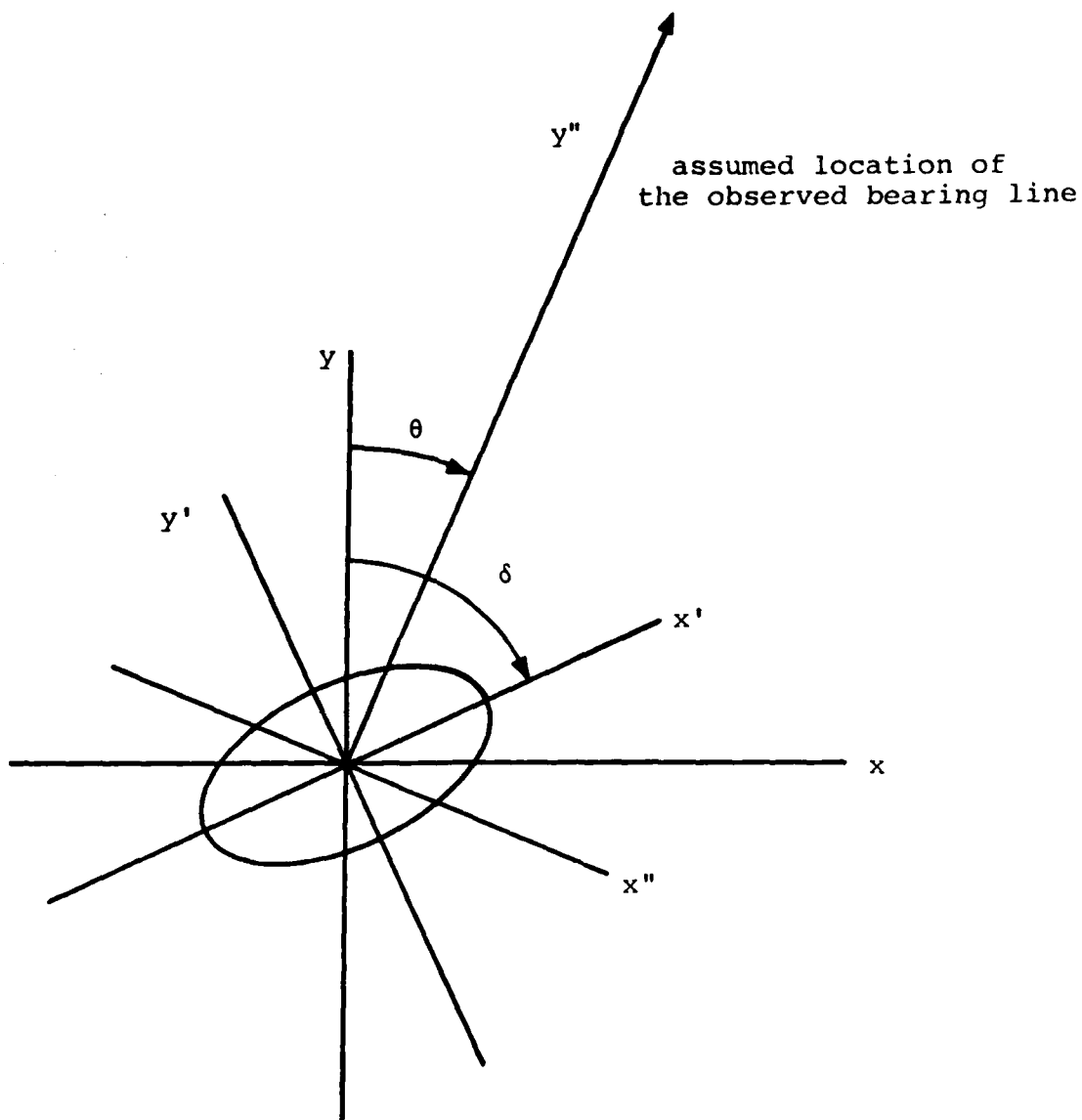
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Figure 1. The geometry associated with the determination of σ_s, the standard deviation of the normal distribution that determines the distance between the assumed location of an observed bearing line and the observed bearing line. The assumed (mean) position of the station is at the origin. The ellipse represents a contour on the probability density surface of the bivariate normal distribution that determines the station's position.

satisfies the condition: $0° \leq \delta < 180°$. The angle θ is an observed bearing. The x"y"-coordinate system is oriented so that the positive y"-axis is in the direction of the observed bearing line. As a consequence of these relationships,

$$\sigma_s^2 = \sigma_{x'}^2 \sin^2(\delta-\theta) + \sigma_{y'}^2 \cos^2(\delta-\theta)$$

where $\sigma_{x'}^2$ and $\sigma_{y'}^2$ are the elements of the station position error covariance matrix relative to the x'y'-coordinate system.

A procedure for determining position estimates that are based on bearings on or from stations at known positions is described in Appendix 1 of Reference 3. The procedure is based on a model that is equivalent to one that is described in Reference 2. The procedure that is described in this section is based on a model that is an extension of it.

The procedure in Appendix 1 of Reference 3 is based on a model in which a station's bearing error is determined by a normal distribution with zero mean (bias) and standard deviation e. The bearing error is related to the distance on a circular arc between a station's true bearing line and observed bearing line. The arc is on the circle with its center at the station that passes through an initial estimate of an object's position. This distance is determined by a normal distribution with mean zero and standard deviation $\sigma = re$ where r is the range of the initial estimate from the station and the standard deviation e is measured in radians. In the model, arc distance

is approximated using a first order approximation which in effect replaces the circle with its tangent line at the initial estimate. As a consequence, the distance u on the tangent line between the observed bearing line and the true bearing line is determined by a normal random variable with mean zero and standard deviation σ. The distance u can be expressed in terms of w, the distance on the tangent line between the observed bearing line and the initial estimate that is also determined by a normal random variable with standard deviation σ, and v, the distance on the tangent line between the true bearing line and the initial estimate. And, as shown in Appendix 1 of Reference 3, this distance can be expressed in terms of the unknown coordinates of the object's position.

The effect of station position uncertainty is accounted for by the distance s between the observed bearing line and the assumed observed bearing line. In the model s is determined by a normal distribution with mean zero and standard deviation σ_s as given above. This approximation is consistent with the first order approximation of arc distance. As a consequence of these two approximations, all of the bearing lines are replaced by lines parallel to the line joining the initial estimate's position and the station's assumed position, both of which are known positions. The geometry involved in shown in Figure 2. The modified relationships resulting from the introduction of station position uncertainty are shown in Figure 3. The
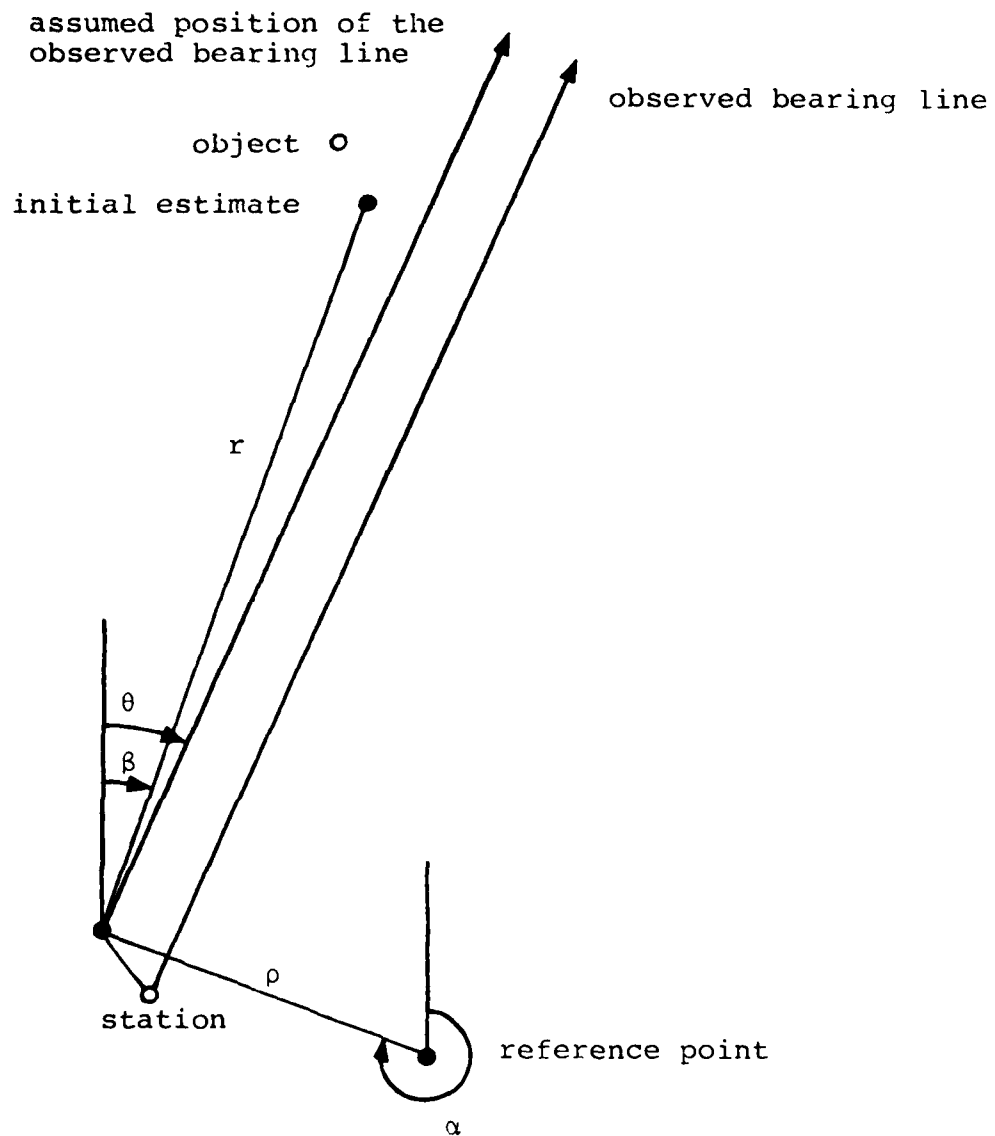
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assumed position of the
observed bearing line

observed bearing line

object O

initial estimate

r

θ

β

ρ

station

reference point

α

Figure 2.   The geometry of the position estimation model.   The
bearing β and range r of the initial estimate from
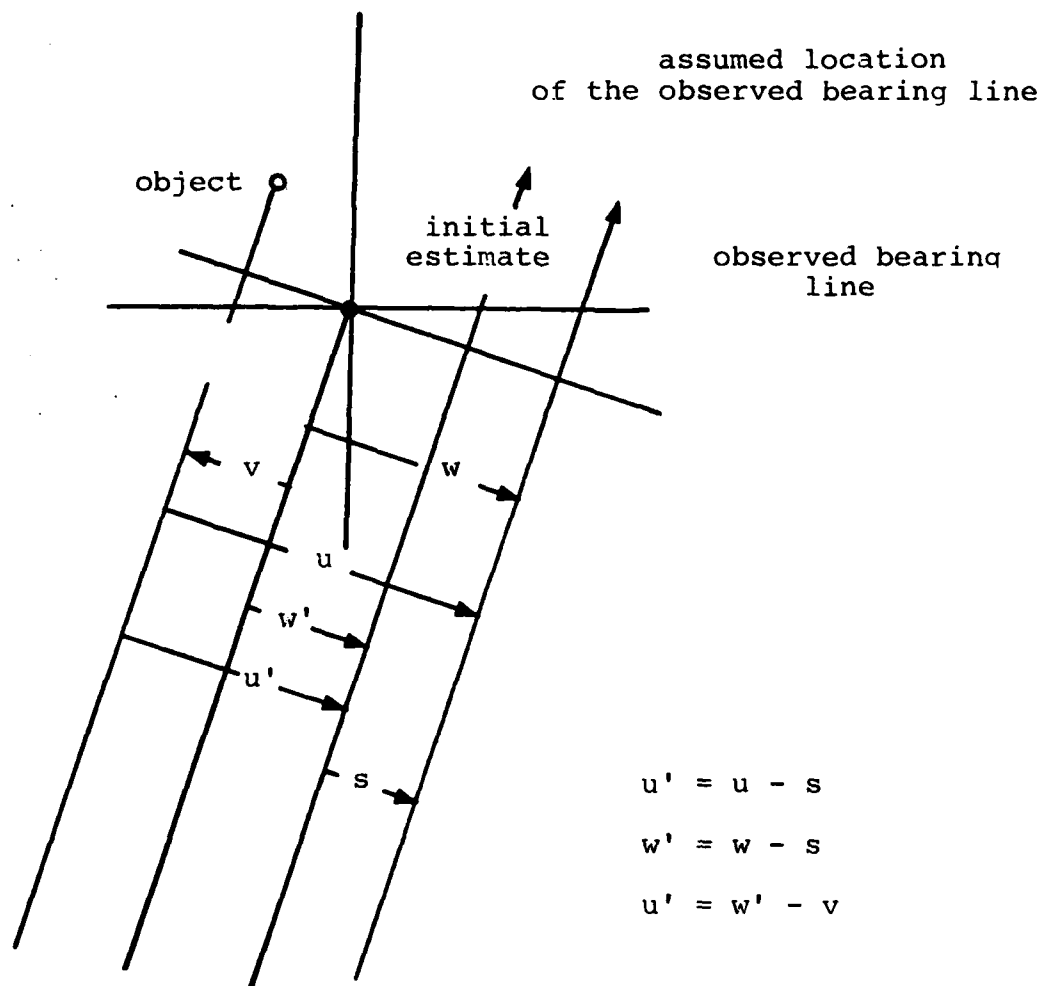the assumed station position have the role of β
and r in Appendix 1 of Reference 3.

Figure 3. The quantities u, v and w correspond to the quantities u, v and w in Appendix 1 of Reference 3. The auxiliary quantities u' and w' are defined in the figure. The replacement of σ^2 by $\sigma^2 + \sigma_s^2$ in the procedure in Appendix 1 of Reference 3 is justified by noting that with this substitution u' and w' are equivalent to u and w.

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important thing to note is that u' = u - s is determined by a
normal distribution with mean zero and standard deviation
$[\sigma^2 + \sigma_s^2]^{\frac{1}{2}}$ and that otherwise U' is equivalent to U with
respect to the procedure in Appendix 1 of Reference 3. As a
consequence of this, the procedure can be extended to include
station uncertainty by replacing σ by $[\sigma^2 + \sigma_s^2]^{\frac{1}{2}}$ where ever
it is used. In this case, σ = re where r is the range of an
initial estimate of an object's position from the assumed station
position and e is the bearing error (standard deviation) in
radians of the bearings associated with the station.

III. Line of Position Uncertainty and Position Estimates

The procedure that is described in this section relates position estimates that are based on lines of position to line of position uncertainty. The procedure is based on a model that is defined as follows: Lines of position are straight lines. Observed lines of position are parallel to true lines of position. The distance of an observed line of position from a true line of position is determined by an independent normally distributed random variable with known mean and standard deviation. Lines of position are specified in terms of a rectangular coordinate system with the origin at a reference point as shown in Figure 4. For celestial navigation, an appropriate choice for the reference point would be the assumed position. Since bearing lines are lines of position, this model differs from the model that is described in Section II only in terminology. However, operationally the use of the model that is described in this section differs in the way the standard deviation of the distance of the line from a true line is determined. The standard deviation associated with each line of position must be specified. If this can be done, the procedure can be used. As an example, suppose that the values are σ_1 for the first line of position and σ_2 for the second where $\sigma_1 > \sigma_2$ and that the lines of position intersect at a 90° angle. In this example, the minimum area confidence (probability) region is an ellipse that is centered on the estimated position. And the
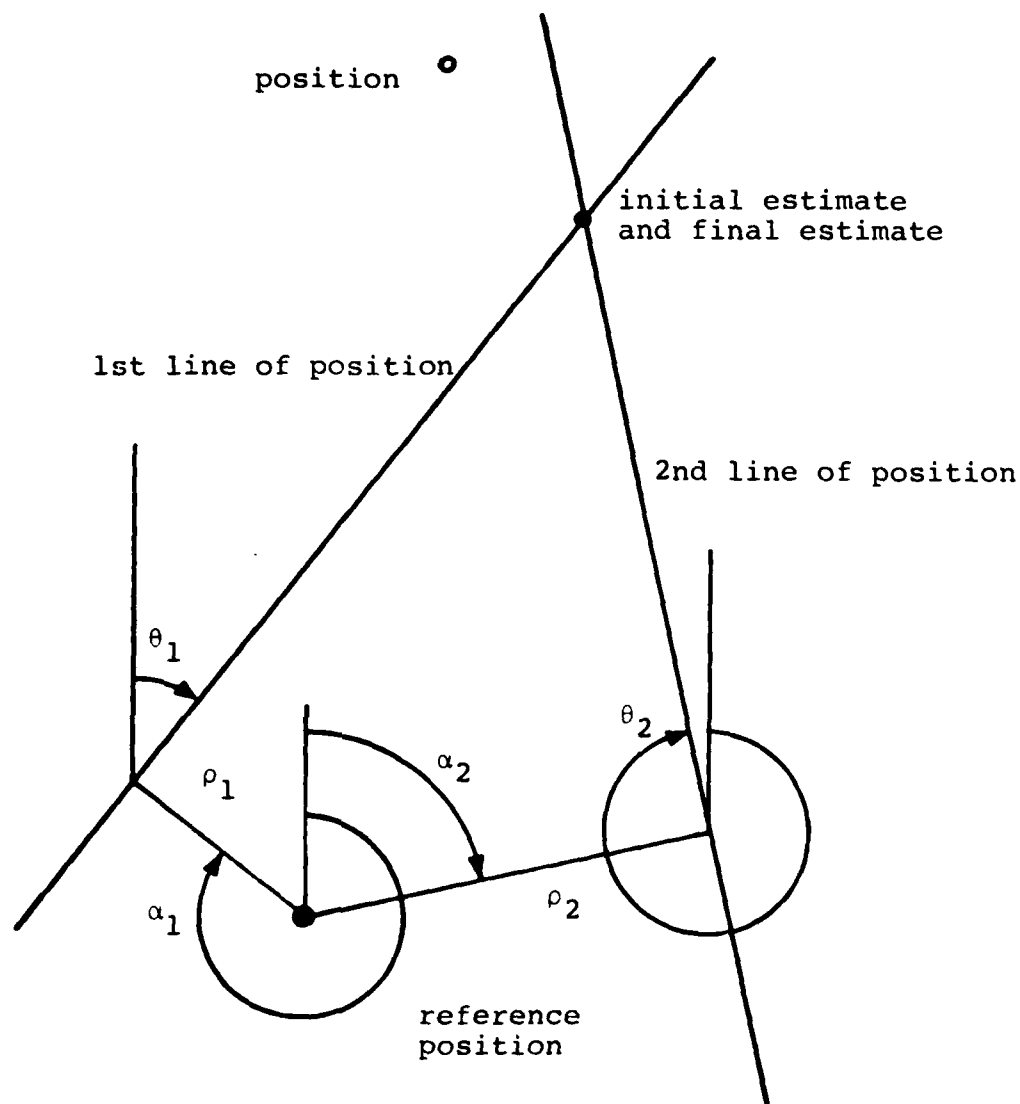
Figure 4. The geometry when only two lines of position are used with the procedure in Appendix 1 of Reference 3. The lines of position correspond to bearing lines for observed bearings θ_1 and θ_2 with respect to the procedure. The procedure determines the size and the orientation of elliptical confidence regions of confidence p, the location of the initial estimate and the location of the final estimate. The locations of the estimates correspond when only two lines are used. If the lines were obtained from sextant observations, the assumed position should be the reference position in which case the lines of position would be determined by the azimuth anges α_1 and α_2 and the distances ρ_1 and ρ_2.

estimated position is at the intersection of the bearing lines. For a confidence (probability) of containment of $1 - \exp(-k^2/2)$, the major axis of the ellipse is coincident with the first bearing line and it is of length $2k\sigma_1$, the minor axis of the ellipse is coincident with the second bearing line and it is of length $2k\sigma_2$ and the area of the ellipse is $\pi k^2 \sigma_1 \sigma_2$.

IV. A Composite Position Estimate

The procedure that is described in this section is for combining position estimates for an object from independent sources. It is based on the following model: The rectangular coordinates of each position estimate are determined by an independent bivariate normal distribution whose covariance matrix is known but whose mean vector is not known. The components of the mean vector for each of the distributions are the unknown coordinates x and y of the object. This model implies that the natural logarithm of the likelihood function for a set of n estimates can be expressed as follows:

$$\log L = K - 1/2 \sum_{i=1}^{n} (\hat{\underline{x}}_i - \underline{x})' \Sigma_i^{-1} (\hat{\underline{x}}_i - \underline{x})$$

where K is a constant, $\hat{\underline{x}}_i$ is an estimate vector with components $\hat{x}_i$ and $\hat{y}_i$, $\underline{x}$ is the common mean vector with components x and y, the unknown coordinates of the object, and Σ_i is the covariance matrix with elements $\sigma_{i\hat{x}}^2$, $\sigma_{i\hat{y}}^2$ and $\sigma_{i\hat{x}\hat{y}}$. The maximum likelihood estimates $\hat{x}$ and $\hat{y}$ of the unknown coordinates x and y are the solutions of the two simultaneous linear equations determined by

$$\frac{\partial(\log L)}{\partial x}\bigg|_{\substack{x=\hat{x}\\y=\hat{y}}} = 0 \quad \text{and} \quad \frac{\partial(\log L)}{\partial y}\bigg|_{\substack{x=\hat{x}\\y=\hat{y}}} = 0 \quad .$$

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The equations can be written as

$$A\,\hat{x} + B\,\hat{y} = D$$

$$B\,\hat{x} + C\,\hat{y} = E$$

and their solution as

$$\hat{x} = \frac{CD - BE}{AC - B^2} \qquad \hat{y} = \frac{AE - BD}{AC - B^2}$$

where $A = \Sigma a_i$, $B = \Sigma b_i$, $C = \Sigma c_i$, $D = \Sigma(a_i\hat{x}_i + b_i\hat{y}_i)$,

$E = \Sigma(b_i\hat{x}_i + c_i\hat{y}_i)$, $a_i = \sigma_{i\hat{y}}^2/d_i$, $b_i = -\,\sigma_{i\hat{x}\hat{y}}/d_i$, $c_i = \sigma_{i\hat{x}}^2/d_i$,

$d_i = \sigma_{i\hat{x}}^2\sigma_{i\hat{y}}^2 - \sigma_{i\hat{x}\hat{y}}^2$ and all of the sums are for i from 1 to n.

Since they are linear combinations of the estimates $\hat{x}_i$ and $\hat{y}_i$, the estimates $\hat{x}$ and $\hat{y}$ are determined by a bivariate normal distribution. Consequently, all that is required to determine this distribution is its mean vector with components $\mu_{\hat{x}}$ and $\mu_{\hat{y}}$ and its covariance matrix with elements $\sigma_{\hat{x}}^2$, $\sigma_{\hat{y}}^2$ and $\sigma_{\hat{x}\hat{y}}$. The mean vector is determined by

$$\mu_{\hat{x}} = E\{(CD - BE)/(AC - B^2)\} = x$$

and

$$\mu_{\hat{y}} = E\{(AE - BD)/(AC - B^2)\} = y$$

And, the covariance matrix is determined by

$$\sigma_{\hat{x}}^2 = E\{(CD - BE) - E(CD - BE)\}^2/(AC - B^2)^2$$

$$= \{C^2(F+I+2H) - 2CB(H+K+M) + B^2(G+J+2K)\}/(AC - B^2)^2$$

$$\sigma_{\hat{y}}^2 = E\{(AE - BD) - E(AE - BD)\}^2/(AC - B^2)^2$$

$$= \{A^2(G+J+2N) - 2AB(H+K+M) + B^2(F+I+2L)\}/(AC - B^2)^2$$

$$\sigma_{\hat{x}\hat{y}} = E\{[(CD-BE)-E(CD-BE)]\cdot[(AE-BD)-E(AE-BD)]\}/(AC - B^2)^2$$

$$= \{(AC+B^2)(H+K+M)-CB(F+I+2L)-BA(G+J+2N)\}/(AC - B^2)^2$$

where $F = \Sigma a_i^2 \sigma_{i\hat{x}}^2$, $G = \Sigma b_i^2 \sigma_{i\hat{x}}^2$, $H = \Sigma a_i b_i \sigma_{i\hat{x}}^2$, $I = \Sigma b_i^2 \sigma_{i\hat{y}}^2$,

$J = \Sigma c_i^2 \sigma_{i\hat{y}}^2$, $K = \Sigma b_i c_i \sigma_{i\hat{y}}^2$, $L = \Sigma a_i b_i \sigma_{i\hat{x}\hat{y}}$, $M = \Sigma (a_i c_i + b_i^2) \sigma_{i\hat{x}\hat{y}}$

and $N = \Sigma b_i c_i \sigma_{i\hat{x}\hat{y}}$ where all of the sums are for $i = 1$ to n.

By using arguments given in Appendix 1 of Reference 3, one can show that the axes of the elliptical confidence regions associated with $\hat{x}$ and $\hat{y}$ are coincident with an x'y'-coordinate system where the transformation from the xy-coordinate system to this system is the coordinate axes rotation through the angle γ defined by $\tan 2\gamma = 2\sigma_{\hat{x}\hat{y}}/(\sigma_{\hat{y}}^2 - \sigma_{\hat{x}}^2)$. For a confidence p, from Reference 3, the minimum area confidence region is an ellipse with semi-axes $k\,\sigma_{\hat{x}'}$ and $k\,\sigma_{\hat{y}';}$, and area $\pi k^2 \sigma_{\hat{x}'} \sigma_{\hat{y}'}$, where $k = [-2 \ln(1-p)]^{\frac{1}{2}}$,

$$\sigma_{\hat{x}'}^2 = \sigma_{\hat{x}}^2 \cos^2\gamma - 2\sigma_{\hat{x}\hat{y}} \cos\gamma \sin\gamma + \sigma_{\hat{y}}^2 \sin^2\gamma,$$

$$\sigma_{\hat{y}'}^2 = \sigma_{\hat{x}}^2 \sin^2\gamma + 2\sigma_{\hat{x}\hat{y}} \cos\gamma \sin\gamma + \sigma_{\hat{y}}^2 \cos^2\gamma$$

and the center of the ellipse is at the point $(\hat{x},\hat{y})$. In this coordinate system, $\sigma_{\hat{x}'\hat{y}'} = 0$.

The above equations can be used to specify a composite position estimate in terms of the location, orientation and size of an elliptical confidence region which is generally the form in which position estimates of the kind that are being considered here are specified. But, since values of $\sigma_{\hat{x}}^2$, $\sigma_{\hat{y}}^2$ and $\sigma_{\hat{x}\hat{y}}$ are required for each of the n estimates that is being combined, a way is needed for determining these values given the orientation and size of an elliptical confidence region. A procedure to do this when the orientation is given in terms of the direction δ of the major axis and the size is given in terms of the lengths SMJ and SMI of the semi-major and semi-minor axes and the confidence p is described next.

By using an xy-coordinate system in which the positive y-axis direction is north and the positive x-axis direction is east and with the convention $0° \leq \delta < 180°$, the dependence of the value of the rotation angle γ and of the order relation between $\sigma_{\hat{x}'}$ and $\sigma_{\hat{y}'}$ on the value of the major axis direction δ is indicated by the following table:

$$0° \leq \delta < 45°: \qquad \gamma = \delta \qquad \text{and } \sigma_{\hat{y}'} > \sigma_{\hat{x}'}$$

$$45° \leq \delta < 135°: \qquad \gamma = \delta - 90° \text{ and } \sigma_{\hat{x}'} > \sigma_{\hat{y}'}$$

$$135° \leq \delta < 180°: \qquad \gamma = \delta - 180° \text{ and } \sigma_{\hat{y}'} > \sigma_{\hat{x}'}$$

With an order relation and a value for p, values for $\sigma_{\hat{x}'}$ and $\sigma_{\hat{y}'}$ can be determined with values for SMJ and SMI. With values for $\sigma_{\hat{x}'}$, $\sigma_{\hat{y}'}$ and γ, values for $\sigma_{\hat{x}}^2$, $\sigma_{\hat{y}}^2$ and $\sigma_{\hat{x}\hat{y}}$ can be found from the following equations:

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$$\sigma_{\hat{x}}^2 = \sigma_{\hat{x}'}^2 \cos^2 \gamma + \sigma_{\hat{y}'}^2 \sin^2 \gamma,$$

$$\sigma_{\hat{y}}^2 = \sigma_{\hat{x}'}^2 \sin^2 \gamma + \sigma_{\hat{y}'}^2 \cos^2 \gamma$$

and

$$\sigma_{\hat{x}\hat{y}} = (\sigma_{\hat{y}'}^2 - \sigma_{\hat{x}'}^2) \sin \gamma \cos \gamma$$

which can be obtained by inverting the equations above for $\sigma_{\hat{x}'}^2$ and $\sigma_{\hat{y}'}^2$.

As an example, suppose the data in the following table represent three independent position estimates:

	$\hat{x}$	$\hat{y}$	δ	SMJ	SMI	k
1st	-3.7	18.1	59°	36	20	2
2nd	11.8	8.4	105°	37	11	2
3rd	0	0	146°	45	23	2

Here, distances are in nautical miles and p = .86 in each case.
For this example with values in square nautical miles:

$$\sigma_{1\hat{x}}^2 = 264.58, \quad \sigma_{1\hat{y}}^2 = 159.42 \text{ and } \sigma_{1\hat{x}\hat{y}} = 98.89$$

$$\sigma_{2\hat{y}}^2 = 321.35, \quad \sigma_{2\hat{y}}^2 = 51.15 \text{ and } \sigma_{2\hat{x}\hat{y}} = -78$$

$$\sigma_{3\hat{x}}^2 = 249.20, \quad \sigma_{3\hat{y}}^2 = 389.30 \text{ and } \sigma_{3\hat{x}\hat{y}} = -173.4$$

These values give the following composite estimate:
$\hat{x} = -2.46$ nautical miles and $\hat{y} = 12.26$ nautical miles.

For this case, $\sigma_{\hat{x}}$ = 16.92 nautical miles, $\sigma_{\hat{y}}$ = 6.21 nautical miles and $\sigma_{\hat{x}\hat{y}}$ = -55.1 square nautical miles. And, for k = 2: SMJ = 33.85 nautical miles, SMI = 12.42 nautical miles and δ = 102°. The composite confidence region and its three component confidence regions are shown in Figure 5.

As a second example, suppose each position estimate is determined by a circular normal distribution. Then $\sigma_{i\hat{x}} = \sigma_{i\hat{y}} = \sigma_i$ and $\sigma_{i\hat{x}\hat{y}} = 0$ for i = 1 to n. In this case, the composite estimate is:

$$\hat{x} = (\Sigma \hat{x}_i / \sigma_i) / (\Sigma 1/\sigma_i) \ , \ \hat{y} = (\Sigma \hat{y}_i / \sigma_i) / (\Sigma 1/\sigma_i) \ ,$$

$$\sigma_{\hat{x}}^2 = n/(\Sigma 1/\sigma_i)^2 \ , \ \sigma_{\hat{y}}^2 = n/(\Sigma 1/\sigma_i)^2 \text{ and } \sigma_{\hat{x}\hat{y}} = 0.$$

In this example, since $\hat{x}$ and $\hat{y}$ are determined by a circular normal distribution, the minimum area confidence regions are circles and orientation is not an issue.

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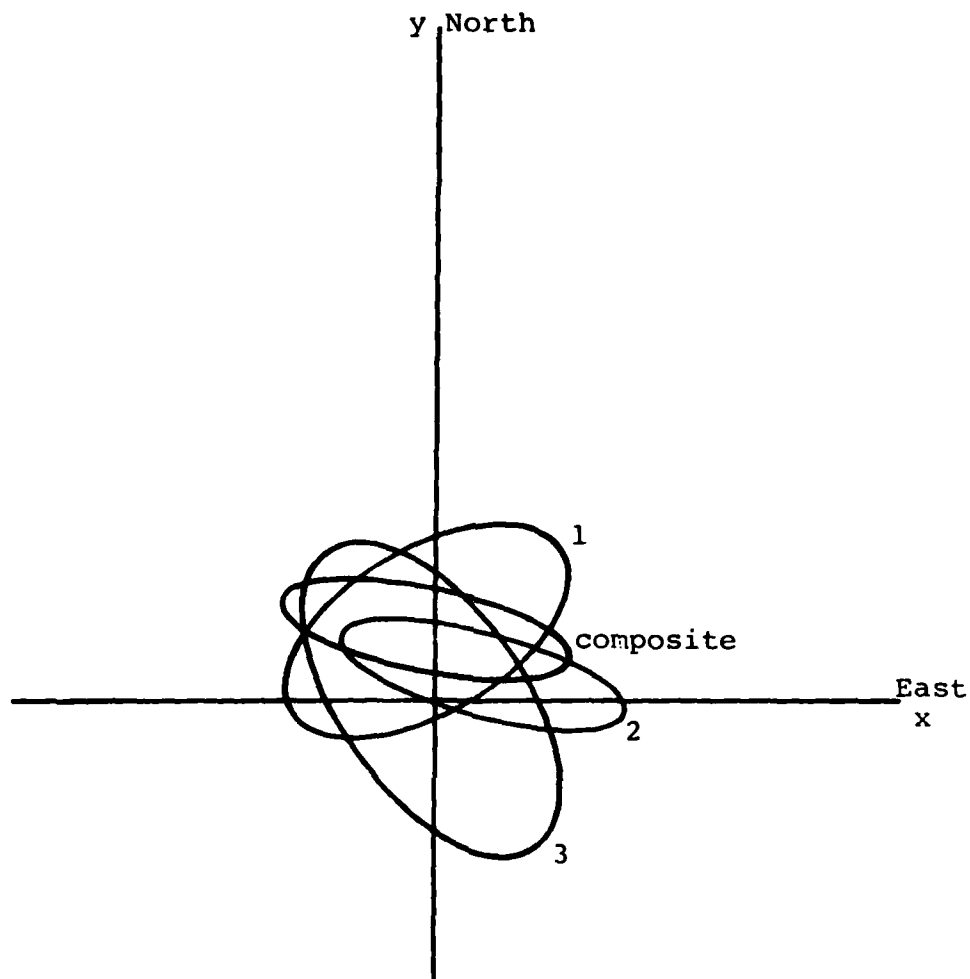
Figure 5. The ellipses represent the confidence ellipses of the first example. The position estimates are at the center of the ellipses. The numbers indicate the order of the estimate in the table on Page 17.

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2. Daniels, H. E., "The Theory of Position Finding," J. Royal Stat. Soc. (B), Vol 13, 1951, pp. 186-207.

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*Note: The angle labeled γ in Figure 4 of this report is the ellipse major axis direction.   The angle between the y and y' axis should be labeled γ for the case indicated in the figure.

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